Tensor Representation and Properties of Multivariable Polynomial

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Abstract: Polynomial theory is the basic of commutative algebra as well as algebraic geometry etc. In addition, it is quite significant in several applied fields. Single variable polynomial theory is almost complete but the structure of multivariable polynomial is not so clear. So, a more concise and simple method is needed. In this paper, tensor representation of polynomial is proved and the validity, algebraic properties and equivalence between tonsorial and traditional representation are shown.

1. The tensor representation of multivariable polynomials

As we all know, the general form of multivariate polynomials on commutative unitary ring is $\sum_{j_1 j_2 \dots j_n} a_{j_1 j_2 \dots j_n} x_1^{j_1} x_2^{j_2} \dots x_n^{j_n}$ (indices will be omitted in the summation symbol), $x_1, x_2 \dots x_n$ are n

indeterminates,

 $j_1, j_2, ..., j_n$ are the degree of indeterminates, $j_1 + j_2 + ... j_n$ is the degree of this monomial, $a_{j_1 j_2 ... j_n}$ represents the coefficient of the term when degree of x_1 is j_1 , degree of x_2 is $j_2 ...$ degree of x_n is j_n . Although such a representation is general, its order and structure are not obvious, and it is not convenient to use, so there are some methods to make its structure clearer, such as lexicographical order, homogeneous multinomial order and so on. Inspired by the fact that symmetric matrices are equivalent to quadratic forms, n-order tensors may also be used. It is equivalent to the "n-degree forms", that is, the n-degrees polynomial. There must be some relationship between them.

Theorem 1: m-variables polynomial $P[x_1, x_2, x_m] = \sum a_{j_1 j_2 \dots j_m} x_1^{j_1} x_2^{j_2} \dots x_m^{j_m}$ and its tensor representation $P[\vec{x}] = A_0 + \vec{A_1} \cdot \vec{x} + \vec{A_2} : (\vec{x} \otimes \vec{x}) + \vec{A_3} : (\vec{x} \otimes \vec{x} \otimes \vec{x}) + \dots + \vec{A_n}^{(n)} \cdot (n) \vec{x}^{\otimes n}$ is equivalent. A_0 is a 0th order tensor, i.e. a constant, $\vec{A_1}$ is an m-dimensional vector, $\vec{A_2}$ is a symmetric matrix, and $\vec{A_n}$ is an n-th order square tensor (That is, the positive integer set of each index is the same); $\vec{x} = (x_1, x_2, \dots, x_m)^T$ is the vector of all indeterminates, \otimes denotes the tensor product, $\vec{x}^{\otimes n} = \vec{x} \otimes \vec{x} \otimes \dots$ (multiply by n times)... $\otimes \vec{x}$, and $\cdot^{(n)}$ means n times dot product.

Proof: $f_n + f_{n-1} + \ldots + f_0$ is obtained by the homogeneous component order for any multivariable polynomial $\sum a_{j_1 j_2 \ldots j_m} x_1^{j_1} x_2^{j_2} \ldots x_m^{j_m}$. f_n refers to the sum of all n-degree monomials, and the internals of each homogeneous component f are lexicographically ordered. We only need to prove that each homogeneous component f is equivalent to a tensor which its order is the same as the degree of f. Then it can be proved that the entire multivariable polynomial is equivalent to the tensor representation.

¹ Notice the number of indeterminates and the degree of the polynomial is different. We use m to represent the dimension of \vec{x} (number of indeterminates) and n means the degree of polynomial. In fact, it is unnecessary to specify dimension of \vec{x} in the second part later.

First look at f_0 , obviously $f_0 = a_{0,0,\dots,0} x_1^0 x_2^0 \dots x_m^0 = A_0$. Look at f_1 , let

 $\overrightarrow{A}_{1} = \left(a_{1,0,0...0}, a_{0,1,0...0}, a_{0,0,1...0}, \dots, a_{0,0,0...1}\right)^{T}, \text{ then } \overrightarrow{A}_{1} \cdot \overrightarrow{x} = a_{1,0,0...0} x_{1} + a_{0,1,0...0} x_{2} + \dots + a_{0,0,0...1} x_{m}$ contains all possible one-degree monomials. Then look at f_{2} , $\left| \text{let } \overrightarrow{A}_{2} = \begin{pmatrix} a_{2,0,0...0} & \frac{a_{1,1,0...0}}{2} & \frac{a_{1,0,1...0}}{2} & \dots & \frac{a_{1,0,0...1}}{2} \\ \frac{a_{1,1,0...0}}{2} & a_{0,2,0...0} & \frac{a_{0,1,1...0}}{2} & \dots & \vdots \\ \frac{a_{1,0,1...0}}{2} & \frac{a_{0,1,1...0}}{2} & \vdots & \vdots & \vdots \\ \dots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{a_{1,0,0..1}}{2} & \dots & \dots & a_{0,0,0...2} \end{pmatrix},$

then $\overrightarrow{A_2}: (\overrightarrow{x} \otimes \overrightarrow{x}) = a_{2,0,0...0} x_1^2 + a_{1,1,0...0} x_1 x_2 + a_{1,0,1...0} x_1 x_3 + ... + a_{0,0,0...2} x_m^2$.

Finally, look at the general case f_l , we know that the degree of any one of these monomials must satisfy $j_1 + j_2 + ..., j_m = l$. From another perspective, the tensor product $\overrightarrow{A_l}^{(l)} \cdot {}^{(l)} \overrightarrow{x}^{\otimes l} = A_{i_l i_2 ... i_l}^{(l)} x_{i_1} x_{i_2} ... x_{i_l}$. x_{i_l} Represents the No. i_l component of the vector $\overrightarrow{x} = (x_1, x_2, ..., x_m)^T$, $i_1, i_2 ... i_l \in \{1, 2..., m\}$.

The first term of the lexicographical order is $a_{l,0,0...0}x_1^l$, and let the component $A_{l,1...1}^{(l)}$ of tensor $\overrightarrow{A_l}^{(l)}$ be $a_{l,0,0...0}$ ($A_{l,1...1}^{(l)} = a_{l,0,0...0}$), when each index $i_1 = i_2 = ... = i_l = 1$. The second term of the lexicographical order is $a_{l-1,1,0...0}x_1^{l-1}x_2$, let the tensor indices take $i_1 = 2$, $i_2 = ... = i_l = 1$ (components $A_{2,1...1}^{(l)}$), $i_2 = 2$, $i_1 = ... = i_l = 1$ (component $A_{1,2...1}^{(l)}$) ...and $i_l = 2$, $i_1 = ... = i_{l-1} = 1$ (component $A_{1,1...2}^{(l)}$), that is, each index takes 2 in order, and the remaining 1. These components satisfy the relationship $A_{2,1...1}^{(l)} + A_{1,2...1}^{(l)} + ... + A_{1,1...2}^{(l)} = a_{l-1,1,0...0}$. For convenience, let all the components at the left-hand side of the equation be equal, get $A_{2,1...1}^{(l)} = A_{1,2...1}^{(l)} = ... = A_{1,1...2}^{(l)} = \frac{a_{l-1,1,0...0}}{l}$. At this time there are

$$A_{2,1\dots 1}^{(l)} x_2 x_1 \dots x_1 + A_{1,2\dots 1}^{(l)} x_1 x_2 \dots x_1 + \dots + A_{1,1\dots 2}^{(l)} x_1 x_1 \dots x_2 = \left(A_{2,1\dots 1}^{(l)} + A_{1,2\dots 1}^{(l)} + \dots + A_{1,1\dots 2}^{(l)}\right) x_1^{l-1} x_2 = a_{l-1,1,0\dots 0} x_1^{l-1} x_2.$$

Observe any $a_{j_1j_2...j_m} x_1^{j_1} x_2^{j_2} ... x_m^{j_m}$ in the homogeneous multinomial sum with degree of 1. We know $j_1 + j_2 + ... j_m = l$, then in $\vec{x} \otimes \vec{x} \otimes ... \otimes \vec{x}$. There must be $j_1 \quad \vec{x}$ (it does not mean j_1 times \vec{x} . It means the number of \vec{x} we choosing is j_1) to select the x_1 component, $j_2 \quad \vec{x}$ to select the x_2 component, $j_m \quad \vec{x}$ to select the x_m component, regardless of order. This can be abstracted into a combinatorial mathematics problem: 1 balls, of which j_1 No. 1 ball, j_2 No.2 ball... j_m No.m ball, how many different arrangements? It is known from the arrangement formula about the repeated collection that their number of permutations is $\frac{l!}{j_1! j_2! ... j_m!}^{[5]}$, therefore, we $(\frac{l!}{j_1! j_2! ... j_m!})$ components in total). And all these components are set to $\frac{j_1! j_2! ... j_m!}{l!} a_{j_1 j_2 ... j_m}$, it is easy to verify the equality of the tensor product expansion and $a_{j_1 j_2... j_m} x_1^{j_1} x_2^{j_2} ... x_m^{j_m} \square$.

So far, we prove that any m-variables polynomial $\sum a_{j_1 j_2 \dots j_m} x_1^{j_1} x_2^{j_2} \dots x_m^{j_m}$ has its tensor representation corresponding to it. In fact, this is not the only one. In the proof process, it can be found that for a monomial $a_{j_1 j_2 \dots j_m} x_1^{j_1} x_2^{j_2} \dots x_m^{j_m}$, there may be many tensors satisfying the condition. Therefore, in order to guarantee the uniqueness of the coefficient tensor, we specify that $A_0, \overrightarrow{A_1}, \dots, \overrightarrow{A_n}^{(n)}$ are symmetric tensors. It means any component $A_{i_1 j_2 \dots j_n}^{(n)}$ stays the same when the arbitrary permutation acting on the indices $i_1 i_2 \dots i_n$. This achieves a one-to-one correspondence with the traditional representation of the m-variables polynomial. On the other hand, we find each item in the tensor representation $\overrightarrow{A_i}^{(i)} \cdot (i) \overrightarrow{x}^{\otimes i}$ corresponds to the sum of all i-degree homogeneous monomials in the traditional notation, that is, i-degree homogeneous components.

2. Property

We have proved that the tensor notation is completely equivalent to multivariable polynomial in the traditional notation, then the various algebraic properties of the multivariable polynomial ring are also inevitably true for the tensor form. The following will explain them by two aspects --additive and multiplication, here its existence and uniqueness will not be described again.

For additive, since only the same order tensors can be added, and the sum and difference of the symmetric tensors are still symmetric tensors, there is

$$\begin{pmatrix} A_0 + \vec{A}_1 \cdot \vec{x} + \vec{A}_2 : (\vec{x} \otimes \vec{x}) + \vec{A}_3 : (\vec{x} \otimes \vec{x} \otimes \vec{x}) + \dots + \vec{A}_m^{(m)} \cdot \stackrel{(m)}{} \vec{x}^{\otimes m} \end{pmatrix} + \\ \begin{pmatrix} B_0 + \vec{B}_1 \cdot \vec{x} + \vec{B}_2 : (\vec{x} \otimes \vec{x}) + \vec{B}_3 : (\vec{x} \otimes \vec{x} \otimes \vec{x}) + \dots + \vec{B}_n^{(n)} \cdot \stackrel{(n)}{} \vec{x}^{\otimes n} \end{pmatrix} \\ = (A_0 + B_0) + (\vec{A}_1 + \vec{B}_1) \cdot \vec{x} + (\vec{A}_2 + \vec{B}_2) : (\vec{x} \otimes \vec{x}) + (\vec{A}_3 + \vec{B}_3) : (\vec{x} \otimes \vec{x} \otimes \vec{x}) \\ + \dots + (\vec{A}_m^{(m)} + \vec{B}_m^{(m)}) \cdot \stackrel{(m)}{} \vec{x}^{\otimes m} + \dots + \vec{B}_n^{(n)} \cdot \stackrel{(n)}{} \vec{x}^{\otimes n}$$

(assuming n > m). That is, the coefficient tensors of the same order are added.

Multiplication is more complicated. Let us first consider $\overleftarrow{A_m}^{(m)} \cdot \overset{(m)}{x} \overrightarrow{x}^{\otimes m}$ and $\overleftarrow{B_n}^{(n)} \cdot \overset{(n)}{x} \overrightarrow{x}^{\otimes n}$ is multiplied, then the other terms are easy to solve. If you directly define the multiplication as $(\overrightarrow{A_m}^{(m)} \otimes \overrightarrow{B_m}^{(m)}) \cdot \overset{(m+n)}{x} \overrightarrow{x}^{\otimes m+n}$, because the result of the tensor product of two symmetric tensors is not necessarily symmetric, the operation is not closed, so we can add one more step, and then symmetrize the tensor product. This operation is defined as the "multiplication" of the operation in the polynomial tensor representation.

We use a concrete example to illustrate the necessity of this step, such as multiplication of a binary quadratic monomial and a binary linear monomial.

$$(ax^{2} + 2bxy + cy^{2})(dx + ey) = adx^{3} + (ae + 2bd)x^{2}y + (2be + cd)xy^{2} + cey^{3}$$
 is written as tensor
notation $(x, y) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \cdot (d, e) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \otimes \begin{pmatrix} d \\ e \end{pmatrix} \vdots \begin{pmatrix} x \\ y \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix}$, because the third-order
tensor can no longer be written on the 2D paper, we should write all the components, so the above

tensor can no longer be written on the 2D paper, we should write all the components, so the above formula is equal to

$$\left(a\left|11\right\rangle + b\left|12\right\rangle + b\left|21\right\rangle + c\left|22\right\rangle\right) \otimes \left(d\left|1\right\rangle + e\left|2\right\rangle\right) \stackrel{\text{\tiny (S)}}{:} \begin{pmatrix}x\\y\end{pmatrix}^{\text{\tiny (S)}}$$
$$= \left(ad\left|111\right\rangle + ae\left|112\right\rangle + bd\left|121\right\rangle + be\left|122\right\rangle + bd\left|211\right\rangle + be\left|212\right\rangle + cd\left|221\right\rangle + ce\left|222\right\rangle\right) \stackrel{\text{\tiny (S)}}{:} \begin{pmatrix}x\\y\end{pmatrix}^{\text{\tiny (S)}}$$

$$= \overleftrightarrow{H}_{3} : \begin{pmatrix} x \\ y \end{pmatrix}^{\otimes 3}.$$

The result is correct, but the coefficient tensor $\overrightarrow{A_3}$ which is dot product with $\begin{pmatrix} x \\ y \end{pmatrix}^{\otimes 3}$ is not symmetric, so we follow the previous statement and add one step to make $\overrightarrow{A_3}$ symmetric, that is, the individual component A_{ijk} of $\overrightarrow{A_3}$ are summed by any arrangement of their indices (a total of 3!) and divided by 3!. So, we get the symmetrized

$$\vec{A}_{3} = ad |111\rangle + \frac{1}{6} (2ae + 4bd) |112\rangle + \frac{1}{6} (2ae + 4bd) |121\rangle + \frac{1}{6} (2cd + 4be) |122\rangle + \frac{1}{6} (2ae + 4bd) |211\rangle + \frac{1}{6} (2cd + 4be) |212\rangle + \frac{1}{6} (2cd + 4be) |221\rangle + ce |222\rangle.$$
 It is easy to

verify that the result is unchanged, and $\overrightarrow{A_3}$ is also symmetric. We put this the defined operation is also recorded as \times .

For a multiplication of m-order monomial and an n-order monomial, there is

$$\left(\overrightarrow{\mathbf{A}_{m}}^{(m)}\cdot^{(m)}\vec{\mathbf{x}}^{\otimes m}\right) \cdot \left(\overrightarrow{\mathbf{B}_{n}}^{(n)}\cdot^{(n)}\vec{\mathbf{x}}^{\otimes n}\right) = \left(\overrightarrow{\mathbf{A}_{m}}^{(m)}\times\overrightarrow{\mathbf{B}_{n}}^{(n)}\right) \cdot^{(m+n)}\vec{\mathbf{x}}^{\otimes m+n} = \operatorname{sym}\left(\overrightarrow{\mathbf{A}_{m}}^{(m)}\otimes\overrightarrow{\mathbf{B}_{n}}^{(n)}\right) \cdot^{(m+n)}\vec{\mathbf{x}}^{\otimes m+n}$$

$$= \frac{1}{(m+n)!} \left(\Sigma_{\sigma(i_{1}i_{2}\dots i_{m}i_{m+1}i_{m+2}\dots i_{m+n})}A_{i_{1}i_{2}\dots i_{m}}^{(m)}B_{i_{m+1}i_{m+2}\dots i_{m+n}}^{(m)}\right) x_{i_{1}}x_{i_{2}}\dots x_{i_{m}}x_{i_{m+1}}\dots x_{i_{m+n}}$$

The symbol $\sum_{\sigma(i_1i_2...i_mi_{m+1}i_{m+2}...i_{m+n})}$ represents the sum of all possible permutation of $i_1i_2...i_mi_{m+1}i_{m+2}...i_{m+n}$. So the product of two polynomials is

$$\begin{pmatrix} A_0 + \vec{A}_1 \cdot \vec{x} + \vec{A}_2 : (\vec{x} \otimes \vec{x}) + \dots \end{pmatrix} \begin{pmatrix} B_0 + \vec{B}_1 \cdot \vec{x} + \vec{B}_2 : (\vec{x} \otimes \vec{x}) + \dots \end{pmatrix}$$

$$= A_0 B_0 + \begin{pmatrix} A_0 \vec{B}_1 + B_0 \vec{A}_1 \end{pmatrix} \cdot \vec{x} + \begin{pmatrix} A_0 \vec{B}_2 + \vec{A}_1 \times \vec{B}_1 + B_0 \vec{A}_2 \end{pmatrix} : (\vec{x} \otimes \vec{x}) + \dots$$

$$= C_0 + \vec{C}_1 \cdot \vec{x} + \vec{C}_2 : (\vec{x} \otimes \vec{x}) + \vec{C}_3 : (\vec{x} \otimes \vec{x} \otimes \vec{x} \otimes \vec{x}) + \dots, \quad \vec{C}_k = \sum_{i+j=k} \vec{A}_i \times \vec{B}_j, \quad k \in \mathbb{Z}.$$

We illustrate the commutation of this multiplication operation. For the monomial operation, since

$$\operatorname{sym}\left(\overrightarrow{\boldsymbol{A}_{m}}^{(m)}\times\overrightarrow{\boldsymbol{B}_{n}}^{(n)}\right)\cdot^{(m+n)}\overrightarrow{\boldsymbol{x}}^{\otimes m+n} = \frac{1}{(m+n)!}\left(\Sigma_{\sigma(i_{1}i_{2}\ldots i_{m}i_{m+1}i_{m+2}\ldots i_{m+n})}A_{i_{1}i_{2}}^{(m)}A_{i_{1}i_{2}\ldots i_{m}}^{(m)}B_{i_{m+1}i_{m+2}\ldots i_{m+n}}^{(n)}\right)x_{i_{1}}x_{i_{2}}\ldots x_{i_{m}}x_{i_{m+1}}\ldots x_{i_{m+n}},$$

by definition, any component of $\overleftarrow{A_m}^{(m)}$ and $\overleftarrow{B_n}^{(n)}$ is an element in \mathbb{R} , the commutative law between components is obviously satisfied, and $i_1i_2...i_mi_{m+1}i_{m+2}...i_{m+n}$ are just indices, so

$$\frac{1}{(m+n)!} \Big(\sum_{\sigma(i_1 i_2 \dots i_m i_{m+1} i_{m+2} \dots i_{m+n})} B_{i_1 i_2 \dots i_n}^{(n)} A_{i_{n+1} i_{n+2} \dots i_{n+m}}^{(m)} \Big) x_{i_1} x_{i_2} \dots x_{i_m} x_{i_{m+1}} \dots x_{i_{m+n}} = \operatorname{sym}\left(\overleftrightarrow{B_n}^{(n)} \times \overleftrightarrow{A_m}^{(m)} \right) \cdot \stackrel{(m+n)}{\longrightarrow} \vec{x}^{\otimes m+n}$$

So, it is easy to know that the polynomial also satisfies the commutative law.

The multiplicative associative law is verified below. Since

$$f = A_0 + A_1 \cdot \vec{x} + A_2 : (\vec{x} \otimes \vec{x}) + A_3 : (\vec{x} \otimes \vec{x} \otimes \vec{x}) + \dots$$

$$g = B_0 + \vec{B}_1 \cdot \vec{x} + \vec{B}_2 : (\vec{x} \otimes \vec{x}) + \vec{B}_3 : (\vec{x} \otimes \vec{x} \otimes \vec{x}) + \dots$$

$$h = C_0 + \vec{C}_1 \cdot \vec{x} + \vec{C}_2 : (\vec{x} \otimes \vec{x}) + \vec{C}_3 : (\vec{x} \otimes \vec{x} \otimes \vec{x}) + \dots \text{ the m-th term of } (fg)h \text{ is}$$

$$\sum_{l+k=m} \left(\sum_{i+j=l} \vec{A}_i \times \vec{B}_j\right) \times \vec{C}_k = \sum_{i+j+k=m} \vec{A}_i \times \vec{B}_j \times \vec{C}_k \text{ , the m-th term of } f(gh) \text{ is}$$

$$\sum_{i+n=m} \overleftarrow{A_i} \times \left(\sum_{j+k=n} \left(\overleftarrow{B_j} \times \overleftarrow{C_k} \right) \right) = \sum_{i+j+k=m} \overleftarrow{A_i} \times \overleftarrow{B_j} \times \overleftarrow{C_k} \text{ . They are consistent.}$$

As for the distribution law, between the tensors having the same order, $\left(\overrightarrow{A_n}^{(n)} + \overrightarrow{B_n}^{(n)}\right) \times \overrightarrow{C_n}^{(n)}$, their components satisfy

$$\begin{aligned} &\frac{1}{(2n)!} \Sigma_{\sigma(i_{1}i_{2}\dots i_{n}i_{n+1}i_{n+2}\dots i_{2n})} \left(A^{(n)}_{i_{1}i_{2}\dots i_{n}} + B^{(n)}_{i_{1}i_{2}\dots i_{n}} \right) C^{(n)}_{i_{n+1}i_{n+2}\dots i_{2n}} \\ &= \frac{1}{(2n)!} \Sigma_{\sigma(i_{1}i_{2}\dots i_{n}i_{n+1}i_{n+2}\dots i_{2n})} \left(A^{(n)}_{i_{1}i_{2}\dots i_{n}} C^{(n)}_{i_{n+1}i_{n+2}\dots i_{2n}} + B^{(n)}_{i_{1}i_{2}\dots i_{n}} C^{(n)}_{i_{n+1}i_{n+2}\dots i_{2n}} \right) \end{aligned}$$

Therefore, the law of distribution is also established. This explains all the properties that must be satisfied for the multivariable polynomial ring.

References

[1] Gu Pei. National Excellent Course - Abstract Algebra [OL]. http://course.jingpinke.com/details?uuid=8a833996-18ac928d-0118-ac928ebc-0156.

[2] Guo Shuohong. Electrodynamics [M]. Beijing: Higher Education Press, 2008.

[3] Deng Shaoqiang, Zhu Fuhai. Abstract Algebra [M]. Beijing: Science Press, 2017.

[4] Zheng XM, the Tensor Series Expansion and Special Functions [J]. China Science & Technology Panorama Magazine, 2017 (5).

[5] Zhang Xiuping. Combinatorial mathematics [M]. Beijing: Beijing Normal University Publishing House, 2011.